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The Application of Non-linear Bi-level Programming to the Aluminium Industry

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Abstract. In this paper, a solution algorithm is presented for the bi-level non-linear programming model developed to represent the complete operations of an aluminium smelter. The model is based on the Portland Aluminium Smelter, in Victoria, Australia and aims at maximising the aluminium production while minimising the main costs and activity associated with the production of this output. The model has two variables, the power input measured in kilo-Amperes (kA) and the setting cycle (of the anode replacement [SC]). The solution algorithm is based on the vertex enumeration approach and uses a specially developed grid search algorithm. An examination of the special nature of the model and how this assists the algorithm to arrive at an optimal unique solution (where there exists one) is undertaken. Additionally, future research into expansion of the model into a multi-period one (i.e., in effect a "staircase" model) allowing the optimisation of the smelter operations over a year (rather than as is currently the case, one month) and the broadening of the solution algorithm to deal with a more general problem, are introduced.

Keywords: Global Optimisation, Bi-level Non-linear Programming, Vertex Enumeration, Grid Search Algorithms

1. Introduction

The mathematical model of the smelter at Portland has been discussed in considerable detail in Nicholls and Hedditch(1993), Nicholls(1993), Nicholls(1994) and Nicholls(1995). In essence, the simple model developed allows the reduction cell, ingot mill and rodding (anode preparation) operations to be represented together with the anode plant production activities. Additionally, the model is able to encapsulate the feed back loop associated with the spent anodes.

The plant operates in such a way that each of the "areas" mentioned above, are autonomous business units within the plant and are able to make decisions in their own right. They are subject to the overriding consideration of maximising aluminium production (essentially the province of the potrooms where the reduction cells are located). However, the operations of anode production and preparation are driven by another variable, the setting cycle (i.e., how long the anodes are left in the reduction cells before replacement). The longer the setting cycle, the less anodes are required per period of time and thus the less these areas need to work and incur costs. Consequently, the potrooms and ingot mill are in effect maximising the production of aluminium, while the anode production and anode preparation areas are concerned with maximising the setting cycle. To this end the model is a bi- level one, as it meets all the requirements as summarised by Wen and Hsu(1991, p 126). The problem is not one belonging to a bi-criteria class even though the traditional solution approaches for the bi-level problem are inapplicable and the problem specification has been mistaken by some for a bi- criteria problem. This classification aspect will be discussed further below.

Because of the special nature of the problem (which is discussed below), the usual approach adopted to solving a linear bi-level programming problem (BLP) of this type (see for example Bard(1983), Anandalingham(1988) and Narula and Nwosu(1985)) is inapplicable. The non-linearity associated with the model together with the unique specification of the objective functions requires a new solution algorithm. With the non-linear bi-level solution algorithm, the basis is also laid for its extension to multivariable problems (i.e., greater than two variables). The solution algorithm developed is also, it is believed, a first in terms of its non-linear capabilities.

In the section below, the mathematical model is depicted and compared to more conventional bi-level problems (and their solution algorithms) to highlight the special nature of the model of the smelter.

2. The Mathematical Model

2.1. The Mathematical Model to the Standard Linear BLP Problem

The solution of non-linear multi-level or bi-level programming problems (NLBLPP) has not seen the same level of success as that enjoyed by linear multi-level programming problems. Generally speaking, solution of the NLBLPP has been very problem (or situation) specific, with a generalised NLBLPP solution algorithm far from a reality. As an example of the "specialist" approach see Khayyal, Horst and Pardalos(1992) where a solution is offered for concave objective function subject to quadratic constraints. Other solutions for specific situations include Suh and Kim(1992), Hobbs and Nelson(1992) and Edmunds and Bard(1991).

The general formulation for the NLBLPP is as follows:

$$\text{P1: } \max_x F(x,y) \quad \text{where } y \text{ solves;} \tag{1a}$$

$$\text{P2: } \quad \max_y f(x,y) \tag{1b}$$

$$\text{s.t., } \quad g(x,y) \le r \tag{1c}$$

$$x, y \ge 0 \tag{1d}$$

$$x \in \mathbf{W}^{n_1}; y \in \mathbf{W}^{n_2} \tag{1e}$$

$$g, F \text{ and} f \text{ non-linear.} \tag{1f}$$

In this model (1), we define the feasible region (constraint region) [**S**], reaction set for $y[\mathbf{R}(x)]$ and solution set for y given $x[\mathbf{Y}(x)]$ as follows.

In (1) above, $(x, y) \in \mathbf{W}^n$ and are the decision variables partitioned as follows; the higher level decision maker has control over the decision vector $x \in \mathbf{W}^{n_1}$ and the lower level decision maker has control over $y \in \mathbf{W}^{n_2}$, where $n_1 + n_2 = n$. The objective functions (and some constraints) are assumed to be linear. The two decision makers are also assumed to play a two person (i.e, duopoly) Stackelberg game.

It is also assumed in solving the problem, that the higher level decision maker selects x first and then the lower level decision maker selects the value of y bearing in mind the value chosen for x.

Here the feasible solution space (problem constraint region) is defines as:

$$\mathbf{S} = \{(x, y) \mid g(x, y) \leq r\} \tag{2}$$

and the set of optimal solutions to the inner problem (P2) given x as $\mathbf{Y}(x)$ having solved the non-linear programming problem:

$$\max f'(y) = dy \tag{3a}$$

$$\text{s.t.,} \quad g^1(y) \leq (r - g^2(x)) \tag{3b}$$

The solution for the higher level decision maker will be selected from: $\mathbf{S}_1 = \{x \mid \exists y \in \mathbf{R}(x) \text{ u.t.}, (x, y) \in \mathbf{S}\}$;

The solution for the lower level decision maker will lie in the region:

$$\mathbf{R}(x) = \{y^\star \mid (x, y) \in \mathbf{S} \text{ and } y^\star \in \mathbf{Y}(x)\} \tag{4}$$

Note that any combination of (x, y) is *feasible* if $x \in \mathbf{S}_1$ and $y \in \mathbf{R}(x)$, i.e,. $(x, y) \in \mathbf{R}(x)$. Optimality, due to the nature of the objective functions in the Portland case, is simply attained with the maximization of $F(x, y)$ followed by $f(x, y)$ in $\mathbf{S}$ and subject to the conditions mentioned above. This will be explained later.

2.2. The Mathematical Model of the Non-linear Bi-level Programming Problem (Portland Aluminium Smelter)

The specific model developed to represent the Portland Aluminium Smelter as follows:

$$\text{P1:} \quad \max_{kA} F(kA, 0) = kA \qquad SC \text{ solves;} \tag{5a}$$

$$\text{P2:} \quad \max_{SC} f(0, SC) = SC \tag{5b}$$

$$\text{s.t.,} \quad \mathbf{a}_1 kA \leq \mathbf{b}_1 \tag{5.1}$$

$$\mathbf{a}_2 SC^{-1} \leq \mathbf{b}_2 \tag{5.2}$$

$$a_3 kA^{-1} + a_4 SC \leq 0 \tag{5.3}$$

$$f_1 kA + d_1 SC^{-1} + e_1 kASC - g_1 \leq b_3 \tag{5.4}$$

$$(f_2 kA + d_2 SC^{-1} + e_2 kASC - g_2)(hSC^{-1} + j)^{-1} \geq b_4 \tag{5.5}$$

$$(f_2 kA + d_2 SC^{-1} + e_2 kASC - g_2)(hSC^{-1} + j)^{-1} \leq b_5 \tag{5.6}$$

$$kA_l \leq kA; 0 \leq SC \leq SC_u \tag{5.7}$$

where $\mathbf{a}_1$ is a $(1 \times m)$ vector of coefficients that are potroom related), $\mathbf{a}_2$ is a $(1 \times n)$ vector of coefficients that are anode production and preparation based. The remaining coefficients are scalars. Thus the formulation of the problem has been achieved with a minimum of complexity (two variables) and around 100 constraints associated with m and n.

Note that there are only two variables in this model, the kA representing the higher level decision maker and SC the lower level decision makers variable. Additionally, there are obvious non-linearities associated with the constraints, namely kA^{-1}, SC^{-1}, and the $SCkA$ terms which with the SC^{-1} term in the constraint mean a quadratic solution must be obtained (i.e., we have a very simple set of objective functions with convex, concave, reciprocal and linear constraints). The latter observation arises due to two factors. The existence of functional relationships that involve the two variables i.e., Returned Anodes $= f(SC, kA)$ and the interdependencies of some of the coefficients (f, g, h etc) with the kA variable.

Also, note that the objective functions are special ones in that they include only one variable, the variable associated with that particular decision maker. This makes the solution of the problem easier in that the solution (if it exists) will be unique and global. The requirements for optimality as set out for the BLP problem above are achieved automatically if the solution pair (kA, SC) are feasible (i.e., $(kA, SC) \in \mathbf{S}$). As is normally the case, in order to obtain a solution, kA is maximised first followed by SC's maximisation given the resultant maximum kA (i.e., kA^{max1^*}). These last two observations relating to optimality contribute substantially to the simplicity of the resultant solution algorithm to this problem.

Note further that the objective functions embody a bi- level relationship since there exists leader and follower decision bodies with their own decision spaces. Conflict resolution in this case requires a sequence of hierarchical independent decisions to be made. The interesting attribute associated with the objective functions is the non-existence of the other decision maker's variable in the others objective function. Bard(1983) has pointed out that in the case where the objective functions are of the form:

$$F = ax + by; f = dy$$

that the problems can be solved as either a bi-criteria or bi- level problem. However, Wen and Hsu(1991) have pointed out that this statement is somewhat contentious.

In the specific case dealt with in this paper, the objective functions are of the form:

$$F = ax; f = by$$

intuitively lending considerably more weight to the claim by Bard(1983) that a solution to one of the problems (say bi- level) is also a solution to the other (bi-criteria). Consequently, it is suggested that the solution algorithm developed in this paper will provide a solution to both the problem classifications. Additionally, it is suggested by the current problem formulation that the leader's solution will be near or at its maximum potential while that of the follower's may well be, as suggested by Bard(1983), below its maximum potential.

In (5) above, by maximising the kA, Problem 1 maximises the output of aluminium. However, by maximising the SC, the P2 objective, the activity associated with a major part of the smelter is reduced, and therefore costs of production are also reduced.

The first group of constraints, (5.1) are basically resource and capacity limitations associated with the potrooms and the ingot mill and directly (and proportionally) related to the kA variable. The second group of constraints (5.2) are associated with the anode production and preparation areas of the smelter. These constraints are specifically related to the number of anodes that must be supplied to the potrooms as a result of the adopted setting cycle. The "demand" for the anodes is a function of the inverse of the setting cycle. The two constraints above are exclusive to one or the other of the variables.

The third constraint (5.3) is essentially indicating that

$$SC \leq (-a_3 kA^{-1})/a_4$$

i.e,. an upper limit on the SC imposed (in addition to the exogenously specified one $SC \leq SC_u$) by the kA being used. This is an important constraint which is brought about by the necessity to ensure that a minimum proportion of the anode used in the potroom is returned for reprocessing in the anode area and incorporation into the manufacture of new anodes. This constraint is the first that requires the existence of both the variables.

The fourth constraint (5.4) ensures that the "filler" resource (coke) is not utilised at a level above its maximum availability. This constraint also sees both the variables involved.

The fifth and sixth constraints, (5.5) and (5.6) relate to the minimum and maximum proportions (respectively) of returned anodes that may constitute the new anodes. The solution space to the problem (**S**) is shown in Figure 1.

2.3. The Solution Algorithm to the Portland Model

As indicated, since this is a very specific problem, with non-general attributes, a specific algorithm has been developed to solve it which will not necessarily be applicable to other problems (or even an expanded version of the current problem). Currently, research is underway to generalise the algorithm to allow it to be able to solve a broader range of problems. The solution space (S) for the model in (5) represents the situation at Portland, and this is a very select part of the possible solution space. In Figure 1 below, a quite different problem is seen if the SC_u is allowed to extend considerably to the right. However, for the Portland model, this area is not considered, and Figure 1 epitomises **S** in which the algorithm operates. In later sections a number of generalisations will be considered.

By grouping (or "blocking") the constraints into four categories corresponding to the functional shapes (i.e., **{1}** involving kA only [linear] and SC^{-1} only [reciprocal], **{2}** involving kA^{-1} and SC [reciprocal], **{3}** kA and SC^{-1} [quadratic:concave])

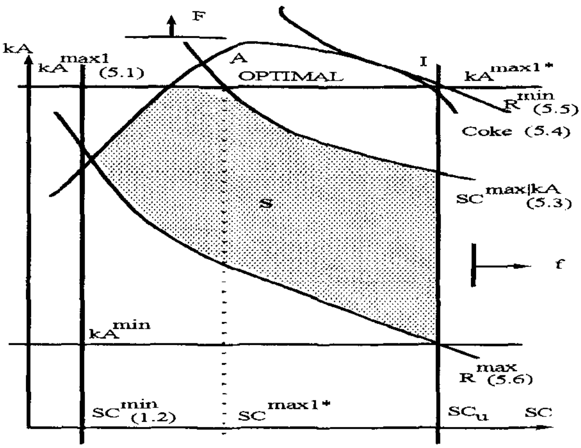

Figure 1. The Solution Space of the Non-Linear Bi-Level Programming Problem

and {4} kA and SC^{-1} [quadratic:convex]) the solution algorithm is able to be developed simply. This blocking is more formal than that adopted by Alexandrov and Dennis(1994), but allows a simpler and more time efficient solution algorithm to be developed. The solution algorithm's simplicity is also contributed to by the nature of the objective functions (each level's objective function involving only that level's variable) and the fact that there are only two variables. If the model had been formulated using the traditional intermediate product modelling approach, then there would have been something like 350 - 400 variables!

Step 1: First Constraint Group - kA Related

In this section of the algorithm, the maximum kA is sought involving the first constraint set (5.1), i.e.,:

$$kA^{max1} = \min_i(b_{1,i}/a_{1,i}) \ (i = 1, \ldots, n) \tag{6}$$

Step 2A: Plant Upper limit to SC

The maximum setting cycle is obtained viz:

$$SC^{max1} = SC_u \tag{7}$$

Step 2B: Second Constraint Group - SC Related

The minimum setting cycle is obtained thus:

$$SC^{min} = \max_{j}(a_{2,j}/b_{2,j}) \ (j = 1, .., m) \tag{8}$$

At this stage the *initial point* (I in Figure 1) has been established which is the intersection of the first two constraint type (5.1) and (5.2) representing the higher and lower level decision makers respectively. This would be the optimal solution if none of the other constraints were to impose themselves on the problem. Thus this point (I) represents the "stepping in" point to the feasible solution space (**S**).

Step 3A: Third Constraint - Maximum SC Given kA

A check is now made with respect to the maximum SC allowed given the current maximum kA (kA^{max1}); If

$$SC^{max} \leq [-a_3/(a_4 kA^{max1})] \tag{9}$$

then the solution obtained at I is still optimal. Proceed to Step 3B.
If (9) is violated, then set $SC^{max} = [-a_3/(a_4 kA^{max1})]$ and proceed to Step 3B. This latter case is illustrated in Figure 1.

Step 3B

A check is then made to ensure that $SC^{max1} \geq SC^{min}$. If this is met then proceed to Step 4 to determine whether the Coke constraint has been met or otherwise. It is clear that the maximum position with respect to kA and SC is being determined via progressive evaluation of the verticies. If $SC^{max1} < SC^{min}$, then the kA must be reduced until the condition is met. This is done via:

$$kA^{max1} = -a_3/(a_4 SC^{min}) \tag{10}$$

where SC^{max1} has been set equal to SC^{min} since the previous SC^{max1} violated SC^{min}. This movement around the solution space is depicted in Figure 2 - Case I.
Assume that the situation above in Step 3B did not eventuate, and that the SC was above the SC^{min}. The next test is to determine whether or not the coke constraint has been violated or not. This is covered in Step 4 below.

Step 4A: Fourth Constraint - Coke Constraint

Evaluate whether the coke constraint has been violated, i.e., see whether

$$f_1 kA^{max1} + d_1/SC^{max1} + e_1 kA^{max1} SC^{max1} - g_1 \leq b_3 \tag{11}$$

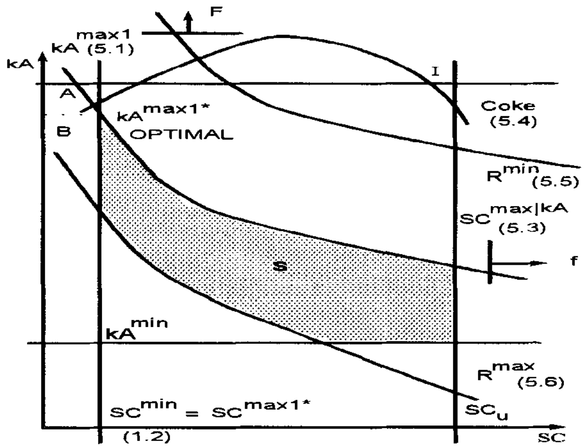

Figure 2. The Solution Space of the Non-Linear Bi-Level Problem - Case I

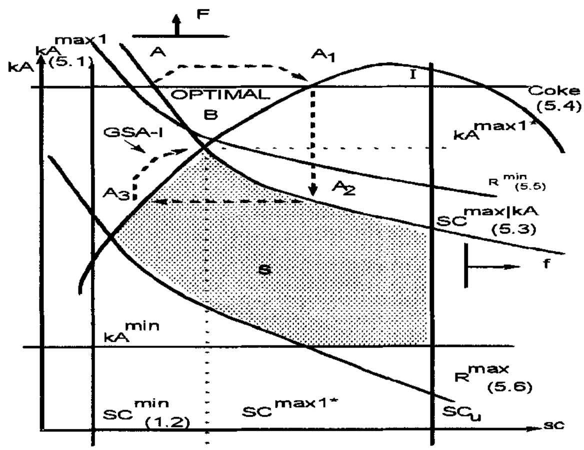

Figure 3. The Solution Space of the Non-Linear Bi-Level Problem - Case III

is met.

If the above constraint is satisfied then proceed to Step 5 below after having first determined that SC that consumes all the available coke with the kA remaining its current value (as per (12) below). This then establishes the SC^{min} if the value of the SC obtained from (12) is smaller that the existing SC^{min} determined in (8) above. If however, the constraint is violated then perform Step 4B. The violation of the coke constraint is depicted in Figure 3 - Case III, together with the other steps required to evaluate another vertex and establish the new solution.

Note that the shape of the coke constraint as depicted in Figure 3 is determined by the parameters of the plant in terms of the net effect on coke of a increase or decrease in the coke consumption compared to the total material needed to make the new number of anodes required.

Step 4B

In this step a move must be effected from A to B but this time in a more complicated way, i.e., along the coke constraint. This is achieved by a grid search algorithm (GSA-I the first of two applications).

The Grid Search Algorithm

Iteration j (note for $j = 1, kA_j = kA^{max1}$ and $SC_j = SC^{max1}$)

1. Determine the SC that will consume all the available coke given the current kA_j (i.e., an increase in the SC) via;

$$SC_j = \max_{k \in R}[(-(f_1 kA_j - g_1 - b_3) \pm ((f_1 kA_j - g_1 - b_3)^2 - 4d_1 e_1 kA_j)^{1/2})(2e_1 kAj)^{-1}] \ (k = 1, 2) \quad (12)$$

   where R is that subset of SC_j that meets the requirements below;

$$SC^{min} \leq SC_j \leq SC_u \quad (12a)$$

   If in (12) $(f_1 kA_j - g_1 - b_3)^2 < 4e_1 kA_j d_1$ is not met, then there is no solution to the problem and the algorithm terminates without considering R. Note if $j > 1$ then this will put the solution at $A3$ and Part 2 will move it to B the optimal solution. Part 1 of the GSA-I has seen the solution shift from A to $A1$ in Figure 3.

2. If R is not empty go to 3. If R is empty, then set $SC_j = SC^{min}$ or SC_u depending on the nature of the violation of (12a) and recalculate the kA_j that causes all the available coke to be used viz;

$$kA_j = (-d_1/SC_j + g_1 + b_3)(f_1 + e_1 SC_j)^-1 \quad (13)$$

   and then proceed to Part 3

3. **If;**

$$SC_j < -a_3/(a_4 kA_j)$$

(which could only occur for $[j = 1]$ when the kA^{max1} and SC^{max1} are unconstrained by the SC^{max} , i.e., at I SC^{max} was greater than the achievable SC or when $[j > 1]$ where the algorithm is set to climb to the optimal solution) then enter the grid search (kA increase phase). Note that if kA is increased a consequential shortage of coke arises since all available coke is already fully used.

Thus;

$$kA_j^* = kA_j + \phi(kA_j - kA_{j-1}), \; 0 < \phi < 1 \tag{14}$$

then set $kA_j = kA_j^*$ and note that for $j = 1$, $kA_0 = kA^{max1}$. A "Return to 1 above" is then affected. Note also that size of ϕ determines the speed of convergence to the optimum (the maximum kA with the SC that causes the constraint to be exactly met). Having ϕ too low will be as inefficient as having it too high. The value of ϕ is experientially set at approximately 0.2 in the Portland Aluminium environment.

If;

$$SC_j = -a_3/(a_4 kA_j)$$

then the search has ended with the maximum kA has been obtained together with the SC being as large as possible (subject to Max kA) with coke fully used. Note that $kA_j \leq kA^{max1}$ (the latter determined in Step 1.) Set $SC^{max1} = SC_j$ and $kA^{max1} = kA_j$ and proceed to Step 5. Note here that $SC^{min} = SC_j$ also.

If;

$$SC_j > -a_3/(a_4 kA_j)$$

(the most likely scenario here for $[j = 1]$) then enter the grid search (kA reduction phase). This is necessary since the current SC exceeds the maximum (given kA_j). The new kA_j is determined thus;

$$kA_j = -a_3/(a_4 SC_j)$$

If $kA_j < kA_l$, set $kA_j = kA_l$ and continue the search by returning to 1 above, however, note that there is now a surplus of coke due to the reduction in kA. This moves the solution from $A1$ to $A2$ in Figure 3, with Part 1 then moving from $A2$ to $A3$. Note that the algorithm is structured so that it will use the coke constraint to climb to the new maximum kA and SC in feasible solution space (with respect to $SC^{max|kA}$ [5.3] and Coke [5.4]).

Figure 4 shows a typical grid search pattern using the grid search algorithm outlined above.

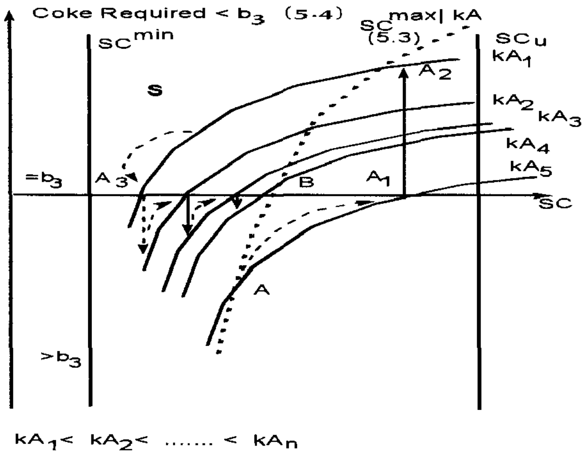

Figure 4. The Grid Search Algorithm - I

Step 5A: Fifth Constraint - Butts Ratio Constraints

This step evaluates whether the ratio of returned anodes to the total anode materials required is within specified limits.

If;

$$(f_2kA^{max1} + d_2/SC^{max1} + e_2kA^{max1}SC^{max1} - g_2)(h/SC^{max1} + j)^{-1} \geq b_4 \quad (15a)$$
$$(f_2kA^{max1} + d_2/SC^{max1} + e_2kA^{max1}SC^{max1} - g_2)(h/SC^{max1} + j)^{-1} \leq b_5 \quad (15b)$$

is met then an optimal solution has been found to the problem as specified in (5) (i.e., $kA^{max1\star}$ and $SC^{max1\star}$ Maximise F and f respectively) and the solution algorithm is terminated.

If the constraints in (15) are not met, then Step 5B is required.

Step 5B

If the *lower limit* of the proportion of anodes returned in the total anode materials (minimum butts ratio) required has not been met, then follow the procedure 1 below. Otherwise go to procedure 2 below.

1. Note that the solution algorithm developed here is based on the observation (at Portland Aluminium), that an increase in the SC will lead to a decrease in the ratio in (15) above i.e., the left side of the ratio function is the operative area ($\partial Ratio/\partial SC < 0$). This due to the fact that the resulting decrease in

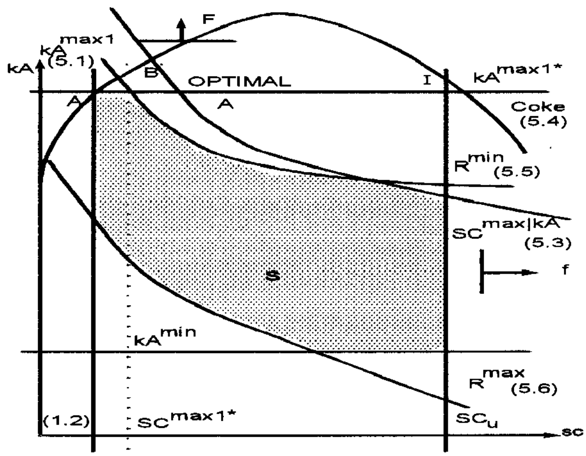

Figure 5. The Solution Space of the Non-Linear Bi-Level Problem - Case IV

total anode material required is proportionally less than the reduction in returned anode material. The alternative is considered later to illustrate how the algorithm can be generalised.

In order to increase the ratio to the required level, the kA^{max1} is left at its current maximum value and the SC that meets the minimum ratio exactly is found via;

$$SC^{max1} = \max_{k \in R}[-(f_2 kA^{max1} - g_2 - b_4 j) \pm ((f_2 kA^{max1} - g_2 - b_4 j)^2 - 4e_2 kA^{max1}(b_4 h_2 - d_2))^{1/2})/2e_2 kA^{max1}] \text{ where, } (k = 1, 2) \tag{16}$$

In (16), R contains the values of the SC that meet the same requirements as laid down for the solution of (14) above.

If R is null, then the same procedure as applied in Step 4B(1) is used, as is the case where R is non-empty (where an optimal solution to the problem may exist). This latter case is shown in Figure 5 - Case IV. Note that here, the coke constraint will now need to be evaluated by returning to Step 4. The above movements are shown in Figure 5 as a move from point A to B.

In the case where R from (16) is null, if SC_j is less than the minimum permitted, then the SC is set equal to the SC^{min} (i.e., $SC^{max1} = SC^{min}$) and a new kA needs to be calculated that will, for the given $SC^{max1} (= SC^{min})$, meet the ratio's lower limit exactly (a move from Point B to C in Figure 6). This is arrived at by using;

$$kA^{max1} = (b_4(h_2/SC^{max1} - j) + g_2 - d_2/SC^{max1})(f_2 + e_2 SC^{max1})^{-1} \tag{17}$$

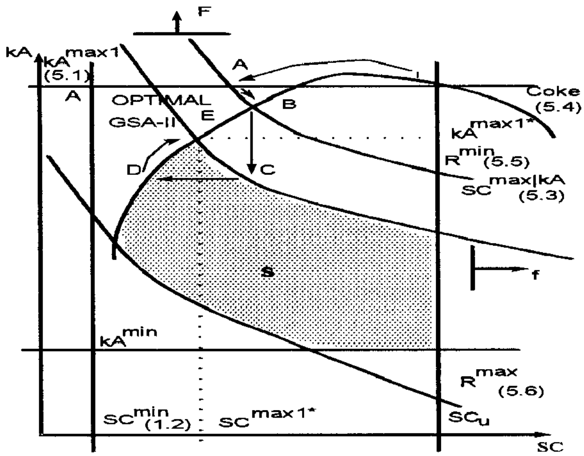

Figure 6. The Solution Space of the Non-Linear Bi-Level Problem - Case V

With the reduction in kA, there is room for the **S** to reduce (point D from in Figure 6). At this point it can be seen that by moving up the coke constraint until the minimum butts ratio is encountered (which is before the $SC^{max|kA}$ is reached) increases in SC and kA are experienced. This is an identical Grid Search Algorithm as that used in the coke violation situation in Step 4. The same procedures are used as encountered in Procedures 2 and 3 in Step 4, with the termination criteria being based on meeting the butts minimum ratio (b_4) rather than $SC^{max|kA}$. This is shown as a movement from point D to E in Figure 6. This second Grid Search Algorithm is designated GSA-II. Note that in this situation, the coke constraint will not be violated since the GSA-II is moving along it. Some general principles are expressed later suggesting why the coke constraint is used here.

2. If the upper limit of the proportion of anodes returned in the total anode materials required has been exceeded, then no solution exists, since the SC and the kA are at their maxima.

3. Extensions To The Solution Space Of The Portland Model

As was indicated previously, the solution space for the Portland Model could be altered slightly, representing somewhat "rare" situations in terms of the smelters configuration and resource availabilities. These however, have not occurred since the plant was commissioned. It is demonstrated below how the algorithm already established, with a few modifications can cope with these changes. Additionally, the solution space is opened further again, and the modifications required to the algorithm indicated. This latter situation however, cannot physically occur, but is demonstrated to show the flexibility of the algorithm developed to solve (5) above, and to foreshadow future work on the algorithm to make is applicable to a broader range of problems. Lastly, the replicated model of the smelter (5) is shown. The solution algorithm for this multivariable model (with perhaps a "cascade" approach to this "staircase" problem [see Jayakumar and Ramasesh(1994)]) is currently under development.

In general terms, Step 1 and 2 of the algorithm are quite clear, they establish the initial absolute limits to **S** and provide a "stepping in" procedure to the solution space proper (**S**). These must be evaluated first.

The next group of constraints to be evaluated are those, involving at least two of the decision variables, and which are capable of intersecting the kA^{max1} established in Step 1 only once. In a truly general algorithm, there would be many constraints evaluated here.

Step 4 is devoted to the evaluation of the quadratic (concave) constraints, which may of course intersect with the kA^{max1} established in Step 1 twice, and will certainly have the ability to intersect with the constraints that intersect kA^{max1} (Step 1) once. ere we are dealing only with the extreme area of **S** consistent with the maximisation of SC and kA, not the lower regions of **S**. The latter would come in to consideration only if it had been violated, and then no feasible solution would exist.

The last Step, deals with the quadratic (convex) constraints which can intersect kA^{max1} established in Step 1 twice also. The reasons for differentiating between the concave and convex constraints is that the derivative sign will be important in terms of the type of grid search direction required. This foreshadows the need to have access to the partial derivatives of the quadratic constraints. The use of these derivatives will be discussed below.

Let the feasible solution space (**S**) be slightly broadened as represented in Figure 7 to include the possibility of meeting the right hand extreme sections of the Coke and Butts Ratio constraints, but not to include any more than the existing number of constraints in these groupings.

Firstly, allow the coke constraint (5.4) to have its maximum well to the left of the SC^{min} as shown in Figure 7. All other constraints are as initially depicted in Figures 1 - 6.

In Figure 7 the initial solution I requires a move to the left in order to meet constraint (5.3) the $SC^{max|kA}$ at point A. However, at A the coke constraint is seen to be violated.(i.e., Step 4A). If the usual Step 4 procedure is used, this will be incorrect since the setting cycle cannot be decreased since it is at the lowest value (SC^{min}) and thus the kA cannot be increased. This Procedure 1 and 2 of Step 4B (i.e., no GSA required) is all that needs to be performed. Thus, from point A existing procedures will take the solution to B which will then be increased to C and the kA dropped until the coke constraint is met exactly. The existing algorithm handles this situation.

If in the above case the minimum butts ratio was violated, then the kA that met the butts ratio exactly with the SC^{min} would be found and the algorithm

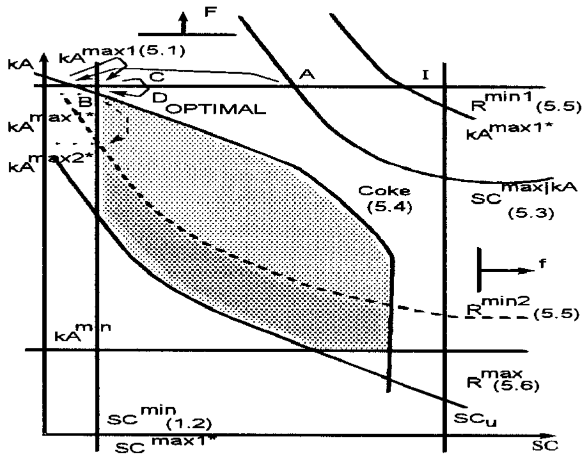

Figure 7. The Solution Space of the Non-Linear Bi-Level Problem - Expanded S

terminated (i.e., Step 5B (17). This would have been the procedure if B had been optimal in Figure 7, provided that the SC^{min} was to the right of the intersection of the minimum butts ratio constraint and the kA^{max1} of Step 1. If the SC^{min} was to the left (or equal) of the intersection of the butts ratio constraint and the kA^{max1} of Step 1B, then the solution would be for the SC^{max1} to reduce to this intersection with kAmax1 not changing.

This truncated solution procedure, requiring no GSA is due to the fact that the active quadratic (concave) constraint has the sum of its partial derivatives with respect to SC and kA, negative. When the sum of the partial derivatives are positive, then the normal solution algorithm (as discussed earlier) is applicable.

The partial derivative of the coke constraint with respect to SC is:

$$\partial C/\partial SC = (-d_1 SC^{-2} + e_1 kA^{max}) \tag{18}$$

while the partial derivative of the coke function with respect to kA is:

$$\partial C/\partial SC = f_1 + e_1 SC \tag{19}$$

The negative sum of the partial derivatives of the coke constraint with respect to SC indicates that the move towards optimality can be primarily attained through the reduction of the setting cycle, with the kA remaining constant (unless certain limits are reached as indicated in Figure 7 (i.e., limits on SC).

It is easy to extend this approach utilising the evaluation of the sum of the partial derivatives on the active constraints to the ratio constraints. However, in the Portland model this is not appropriate since as can be seen from the figures to date, the ratio constraints are not quadratic within the positive quadrant (i.e., within the areas where kA, $SC \leq 0$).

4. An Expansion Of The Portland Model Into A "Staircase" NLBLPP

With the necessity to optimize the smelter over periods of a year or more (on a monthly basis), the model shown in (5) needs to be able to be modified. The months are linked together by various raw and intermediate raw materials, and other intertemporal considerations. The "staircase" involves many variables (i.e., kA_t and SC_t) and the algorithm must now be expanded to allow the solution of this new problem. The modified mathematical model is a follows:

$$\text{P1:} \quad \max_{kA} F(kA_t, 0) = kA_t \text{ where } SC_t \text{ solves;} \tag{20a}$$

$$\text{P2:} \quad \max_{SC} f(0, SC_t) = SC_t \tag{20b}$$

$$\text{s.t.,} \quad \mathbf{a}_{1,t} kA_t \leq \mathbf{b}_{1,t} \tag{20.1}$$

$$\mathbf{a}_{2,t} SC_t^{-1} \leq \mathbf{b}_{2,t} \tag{20.2}$$

$$a_{3,t} kA_t^{-1} + a_{4,t} SC_t \leq 0 \tag{20.3}$$

$$f_{1,t} kA_t + d_{1,t} SC_t^{-1} + e_{1,t} kA_t SC_t - g_{1,t} \leq b_{3,t} \tag{20.4}$$

$$(f_{2,t} kA_t + d_{2,t} SC_t^{-1} + e_{2,t} kA_t SC_t - g_{2,t})(h_t SC_t^{-1} + j_t)^{-1} \geq b_{4,t} \tag{20.5}$$

$$(f_{2,t} kA_t + d_{2,t} SC_t^{-1} + e_{2,t} kA_t SC_t - g_{2,t})(h_t SC_t^{-1} + j_t)^{-1} \leq b_{5,t} \tag{20.6}$$

$$\mathbf{OS}_1 - \mathbf{a}_{1,t} kA_t + \mathbf{DS}_{1,t} - \mathbf{CS}_{1,t} = 0 \tag{20.7}$$

$$\mathbf{OS}_2 - \mathbf{a}_{2,t} SC_t^{-1} + \mathbf{DS}_{2,t} - \mathbf{CS}_{2,t} = 0 \tag{20.8}$$

$$OS_c - b_{3,t} + DS_{c,t} - CS_{c,t} = 0 \tag{20.9}$$

$$kA_{l,t} \leq kA_t; \; 0 \leq SC_t \leq SC_{u,t} \tag{20.10}$$

$$\text{all variables } \geq 0, \; t = (1, ..., T) \tag{20.11}$$

In (20), the $\mathbf{OS}_1$, $\mathbf{OS}_2$, OS_c represent the initial opening stocks of raw materials whose use is, determined by kA, SC and both kA and SC respectively (note that bolded variables represent vectors). Also, $\mathbf{DS}_{1,t}$, $\mathbf{DS}_{2,t}$ and $CS_{c,t}$ represent the deliveries of raw and intermediate materials of the three categories of stock while $\mathbf{CS}_{1,t}$, $\mathbf{CS}_{2,t}$ and $CS_{c,t}$ represent the closing stocks of the three stock categories. The S_t is the hard coded slack variable for the coke constraint.

Note that only for the first period ($t = 1$) will the opening stock variables be used, thereafter they will be replaced by the closing stocks from the previous period ($t-1$), thus producing the "staircase" effect of the model.

There may well be some benefit in considering a "cascade solution" approach here as well, to minimise the solution time.

The aspect of sensitivity analysis has also to be addressed in this model, and research is currently underway here as well.

5. Concluding Remarks

In this paper, a solution algorithm for a NLBLPP has been developed for a specific situation (and thus class of problem). It has been demonstrated that this approach is able to be extended to be applicable for a wider range of problems than the one currently being solved. Extensions to the approach outlined in this paper are currently being researched and tested. The area of NLBLP is a very difficult one with there being no general algorithm currently available, with good reason.

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